<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">Goce Delcev University – Stip, North Macedonia</td></tr>
<tr><td>Topic</td><td colspan="2">Application of Derivatives</td></tr>
<tr><td>Lesson title</td><td colspan="2">Minimizing and Maximizing Problems</td></tr>
<tr><td>Learning objectives</td><td>✓ Students will be able to estimate minimum and maximum values of different sizes using differentiation;
✓ Students will acquire and deal with derivatives of a function;
✓ Students will be able to deal with different problems in everyday life, which require finding minimum or maximum value of a given size;
✓ Students are encouraged to use technology and different software in their work, while considering problem-based situations.</td><td rowspan="3">Strategies/Activities
☐Graphic Organizer
☐Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations
☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics

Assessment as learning
☐Self-assessment
☐Peer-assessment
☐Presentation
☐Graphic Organizer
☐Homework</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to make students able to calculate derivatives of a function and apply the derivatives to calculate minimum and maximum value.

The teacher gives the next problem to the students:

Car B is 30 km directly east of Car A and begins moving west at 90 km/h. At the same moment car A begins moving north at 60 km/h. What will be the minimum distance between the cars and at what time t does the minimum distance occur?</td></tr>
<tr><td>Previous knowledge assumed:</td><td>- formulas for rectilinear motion
- Pythagorean Theorem
- algebraic equations
- differentiating techniques
- chain rule</td></tr>
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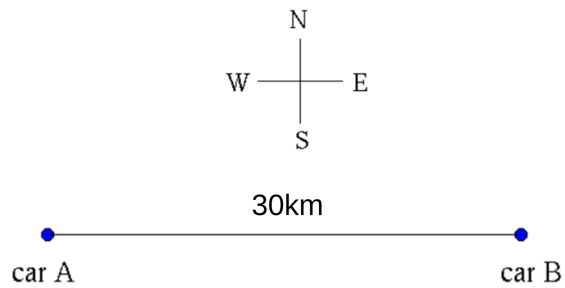
Introduction / Theoretical basics

First, we are considering the given problem.

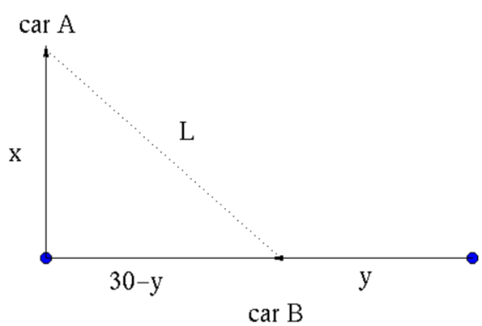


The two cars travel according to the rectilinear motion with the following speeds:

CAR A: North at 60 km/h

CAR B: West at 90 km/h

Let variable x be the distance car A travels in t hours, and variable y the distance car B travels in t hours. Let variable L be the distance between cars A and B after t hours.



Thus, by the Pythagorean Theorem distance L is

$L=\sqrt{x^2+(30-y)^2}$.

If this problem is solving by equations for rectilinear motion is not simple, but derivatives of the functions can help in easily solving of this problem.

Before we differentiate, we will rewrite the right-hand side as a function of t only. Recall that if travel is at a CONSTANT rate then

(distance traveled) = (rate of travel)* (time elapsed)

Assessment of learning

☐Test
☐Quiz
☐Presentation
☐Project
☐Published work

Thus, for car A the distance traveled after t hours is

(Equation 1) x = 60t,

and for car B the distance traveled after t hours is

(Equation 2) y = 90t.

By using of Equations 1 and 2, the equation for L can be rewritten as a function of t only. Thus, we wish to minimize the distance between the two cars:

$$L = \sqrt{x^2 + (30 - y)^2} =$$
$$= \sqrt{(60t)^2 + (30 - 90t)^2} =$$
$$= \sqrt{3600t^2 + (30 - 90t)^2}$$

Such minimizing / maximizing problems can easily be solved with an application of derivatives of a function.

If $y = f(x)$ is given function with domain D and $x_0 \in D$, let $y_0 = f(x_0)$. If the argument x has been changed for Δx and the new one is $x_0 + \Delta x \in D$, then the value of the function changes to $f(x_0 + \Delta x)$.

Definition: If the limit $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x)}{\Delta x}$ exists, we call it *the first derivative of the function* $y = f(x)$ at the point x_0.

According to the definition, the derivative of a function is related to changes of the values of the argument and the function.
If we use function to represent some size, we can use derivative of a function in a problems related to the changes of the values of that size.

Using the definition of the first derivative, the derivatives of some elementary functions are calculated and are now used for calculating derivatives of other functions. A table of some elementary functions with their derivatives is the following:

$$f(x)=c, \quad f'(x)=0$$
$$f(x)=x^n, \quad f'(x)=nx^{n-1}$$
$$f(x)=\frac{1}{x}, \quad f'(x)=-\frac{1}{x^2}$$
$$f(x)=\sqrt{x}, \quad f'(x)=\frac{1}{2\sqrt{x}}$$
$$f(x)=e^x, \quad f'(x)=e^x$$
$$f(x)=\ln x, \quad f'(x)=\frac{1}{x}$$
$$f(x)=a^x, \quad f'(x)=ax^{n-1}$$

Let $f(x)$ and $g(x)$ be differentiable functions and c is a constant. Then the following equations hold:

$(cf(x))'=cf'(x)$ - constant multiple rule

$(f(x)+g(x))'=f'(x)+g'(x)$ - sum rule

$(f(x)-g(x))'=f'(x)-g'(x)$ - difference rule

$(f(x)\cdot g(x))'=f'(x)\cdot g(x)+f(x)\cdot g'(x)$ - product rule

$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{(g(x))^2}$ for $g(x)\neq 0$ - quotient rule

Definition: If $f'(x)$ is a first derivative of a function $f(x)$, then its derivative (if it exists), i.e. $[f'(x)]'=f''(x)$ is called the second derivative of a function $f(x)$.

	The derivatives can be applied for calculating extreme values of different sizes. If $y=f(x)$ is given function, the function f has its minimum value at if $f'(c)=0$ and $f''(c)>0$. The function f has its maximum value at $x=c$ if $f'(c)=0$ and $f''(c)<0.$ If we consider certain size as a function with one variable, we can find its minimum or maximum values with the above rules.	
Action	Let us return to the given problem and construct an appropriate function to find the minimum distance between the two cars. Here is the given problem: *Car B is 30 km directly east of Car A and begins moving west at 90 km/h. At the same moment car A begins moving north at 60 km/h. What will be the minimum distance between the cars and at what time t does the minimum distance occur?* If t is the time, the distance between the two cars at the time t is the function: $$L(t)=\sqrt{3600t^2+(30-90t)^2}\ .$$ According to the rules which determine the extreme values, we have to calculate the first derivative and calculate t such that $L'(t)=0$. In order to find the first derivative, we must use the chain rule.	

$$L'(t)=\frac{7200t-180(30-90t)}{2\sqrt{3600t^2+(30-90t)^2}}=$$

$$=\frac{3600t-2700+8100t}{\sqrt{3600t^2+(30-90t)^2}}=$$

$$=\frac{11700t-2700}{\sqrt{3600t^2+(30-90t)^2}}$$

$$L'(t)=0$$

$$\frac{11700t-2700}{\sqrt{3600t^2+(30-90t)^2}}=0$$

$$11700t-2700=0$$

$$t=\frac{2700}{11700}=0.23$$

We must find the second derivative of the function $L(t)=\sqrt{3600t^2+(30-90t)^2}$ in order to determine is the obtained value maximum or minimum? But this information can be derived from the following assessment: $-\infty$ 0.23 $+\infty$

$L'(t)$	- - -	+ + +

So, at time $t=0.23$ the distance is minimum.

With substitution of the value $t=0.23$ in the distance function $L(t)=\sqrt{3600t^2+(30-90t)^2}$ we obtain the minimum distance

$$L(0.23)=\sqrt{3600\cdot(0.23)^2+(30-90\cdot 0.23)^2}=$$

$$=\sqrt{190.44+86.49}=$$

$$=\sqrt{276.93}=16.64$$

The students can solve similar problems in physics in order to have practical application of their mathematical knowledge from calculus.

Materials / equipment / digital tools / software	Literature given in the references at the end of the document Mathematica for plotting functions.	
Consolidation	With the given examples students can consider that the real functions and their derivatives are important for solving real life problems. Students will learn what is a derivative of a function and how to calculate it. They can learn how to apply differentiation and derivatives to maximize / minimize certain value by given conditions. Students can use technology, different digital tools and software as a help for solving problems, but can also realize that even with technology, solving different everyday problems is difficult without math knowledge.	

Reflections and next steps	
Activities that worked	**Parts to be revisited**
Problem solving, collaboration, using technology	Depends on the students, in a conversation with students the teacher will realize the difficulties that students had and then revisit appropriate parts.

References

[1] M. L. Bittinger, D. J. Ellenbogen and S.A. Surgent (2012), “Calculus and its applications”, Addison-Wesley

[2] G. Strang “Calculus” , Wellelye-Cambridge Press

[3] S. Calaway D. Hoffman and D.Lippman (2014) “Applied Calculus”

[4] *P.D. Lax*, M. S.Terrell (2014) “Calculus with Applications”, Springer